MATH 2230A

Tutorial 1. 14-9-2016.

1) Show that a) $1+z+z^2+\cdots+z^n = \dfrac{1-z^{n+1}}{1-z}$ $(z\neq 1)$

b) (Lagrange's trigonometric identity)

$$1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta = \frac{1}{2}+\frac{\sin\left(\frac{2n+1}{2}\theta\right)}{2\sin\left(\frac{\theta}{2}\right)}$$

Ans: a) $(1-z)(1+z+z^2+\cdots+z^n) = 1+z+z^2+\cdots+z^n-z-z^2-\cdots-z^{n+1}$

$$= 1-z^{n+1}$$

$$\Rightarrow \quad 1+z+z^2+\cdots+z^n = \frac{1-z^{n+1}}{1-z}$$

b) Put $z=e^{i\theta}$ $(\theta\in(0,2\pi))$ into a),

$$1+e^{i\theta}+e^{i(2\theta)}+\cdots+e^{i(n\theta)} = \frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}} \quad \text{——} \quad (*)$$

Recall the Euler formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

In particular we have

$$\begin{cases} e^{i\theta}+e^{-i\theta} = (\cos\theta+i\sin\theta)+(\cos\theta-i\sin\theta) = 2\cos\theta \\ e^{i\theta}-e^{-i\theta} = (\cos\theta+i\sin\theta)-(\cos\theta-i\sin\theta) = 2i\sin\theta \end{cases}$$

Note that
$$\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}} = \frac{e^{i(n+1)\theta}-1}{e^{i\theta}-1}$$
$$= \frac{e^{\frac{i(n+1)\theta}{2}}}{e^{\frac{i\theta}{2}}} \times \frac{e^{\frac{i(n+1)\theta}{2}}-e^{-\frac{i(n+1)\theta}{2}}}{e^{\frac{i\theta}{2}}-e^{-\frac{i\theta}{2}}}$$
$$= e^{\frac{in\theta}{2}} \times \frac{\sin\left(\frac{n+1}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}$$

By considering the real part of $(*)$ on both sides, we have

$$1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta = \cos\left(\frac{n\theta}{2}\right)\frac{\sin\left(\frac{n+1}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}$$
$$= \frac{\sin\frac{\theta}{2}+\sin\left(\frac{2n+1}{2}\theta\right)}{2\sin\frac{\theta}{2}}$$
$$= \frac{1}{2}+\frac{\sin\left(\frac{2n+1}{2}\theta\right)}{\sin\frac{\theta}{2}}$$

2) de Moivre's formula: $\cos n\theta + i\sin n\theta = e^{in\theta} = (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$

For $n=3$, we have

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\ &= (\cos^3\theta - 3\cos\theta\sin\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)\end{aligned}$$

$\Rightarrow \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ & $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

For $n = 2k$, we have

$$\begin{aligned}\cos n\theta + i\sin n\theta &= (\cos\theta + i\sin\theta)^n \\ &= \sum_{j=0}^{n} \binom{n}{j} \cos^{n-j}\theta (i\sin\theta)^j \\ &= \sum_{j=0}^{k} \binom{n}{2j} \cos^{n-2j}\theta (i\sin\theta)^{2j} \\ &\quad + \sum_{j=1}^{k} \binom{n}{2j-1} \cos^{n-2j+1}\theta (i\sin\theta)^{2j-1}\end{aligned}$$

$$\begin{aligned}\Rightarrow \cos n\theta &= \sum_{j=0}^{k} \binom{n}{2j} \cos^{n-2j}\theta (i\sin\theta)^{2j} \\ &= \sum_{j=0}^{k} \binom{n}{2j} (-1)^j \cos^{n-2j}\theta \sin^{2j}\theta\end{aligned}$$

and

$$\sin n\theta = \sum_{j=1}^{k} \binom{n}{2j-1} (-1)^{j-1} \cos^{n-2j+1}\theta \sin^{2j-1}\theta$$

3) Roots of complex number:

If $z = re^{i\theta}$ $(\theta \in (-\pi, \pi])$, then we have

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}, \quad k = 0, 1, 2, \cdots, n-1$$

Find the roots of $8^{\frac{1}{6}}$.

Ans: $8^{\frac{1}{6}} = (8e^{i(0)})^{\frac{1}{6}}$

$$= 8^{\frac{1}{6}} e^{i\left(\frac{0}{6}\right)},\ 8^{\frac{1}{6}} e^{i\left(\frac{0}{6}+\frac{2\pi}{6}\right)},\ 8^{\frac{1}{6}} e^{i\left(\frac{0}{6}+\frac{4\pi}{6}\right)},$$
$$8^{\frac{1}{6}} e^{i\left(\frac{0}{6}+\frac{6\pi}{6}\right)},\ 8^{\frac{1}{6}} e^{i\left(\frac{0}{6}+\frac{8\pi}{6}\right)},\ 8^{\frac{1}{6}} e^{i\left(\frac{0}{6}+\frac{10\pi}{6}\right)}$$

$$= \sqrt{2},\ \frac{1+\sqrt{3}i}{\sqrt{2}},\ \frac{-1+\sqrt{3}i}{\sqrt{2}},\ -\sqrt{2},$$
$$\frac{-1-\sqrt{3}i}{\sqrt{2}},\ \frac{1-\sqrt{3}i}{\sqrt{2}}$$

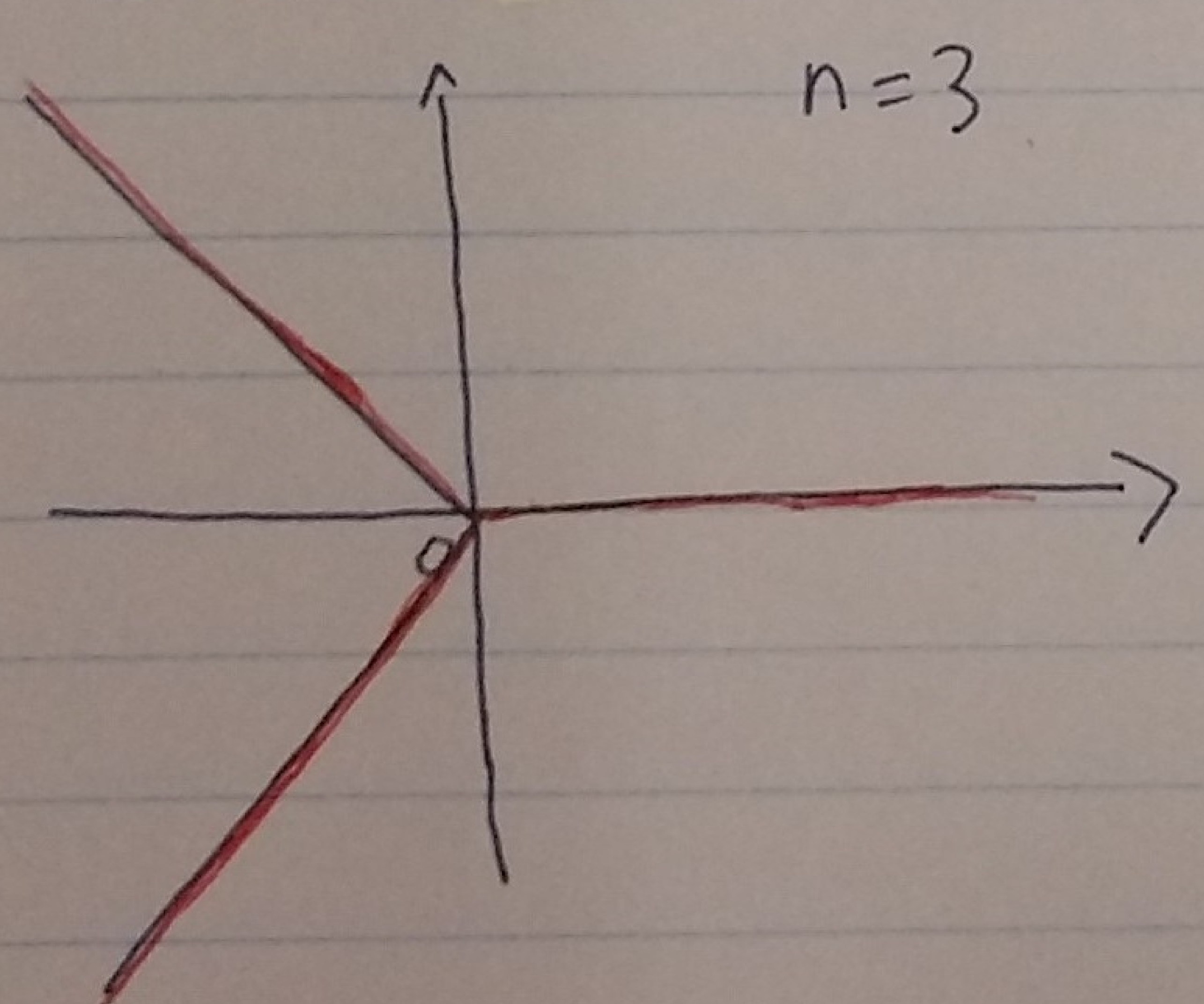
4) Solve the equation $|z|^n = z^n$.

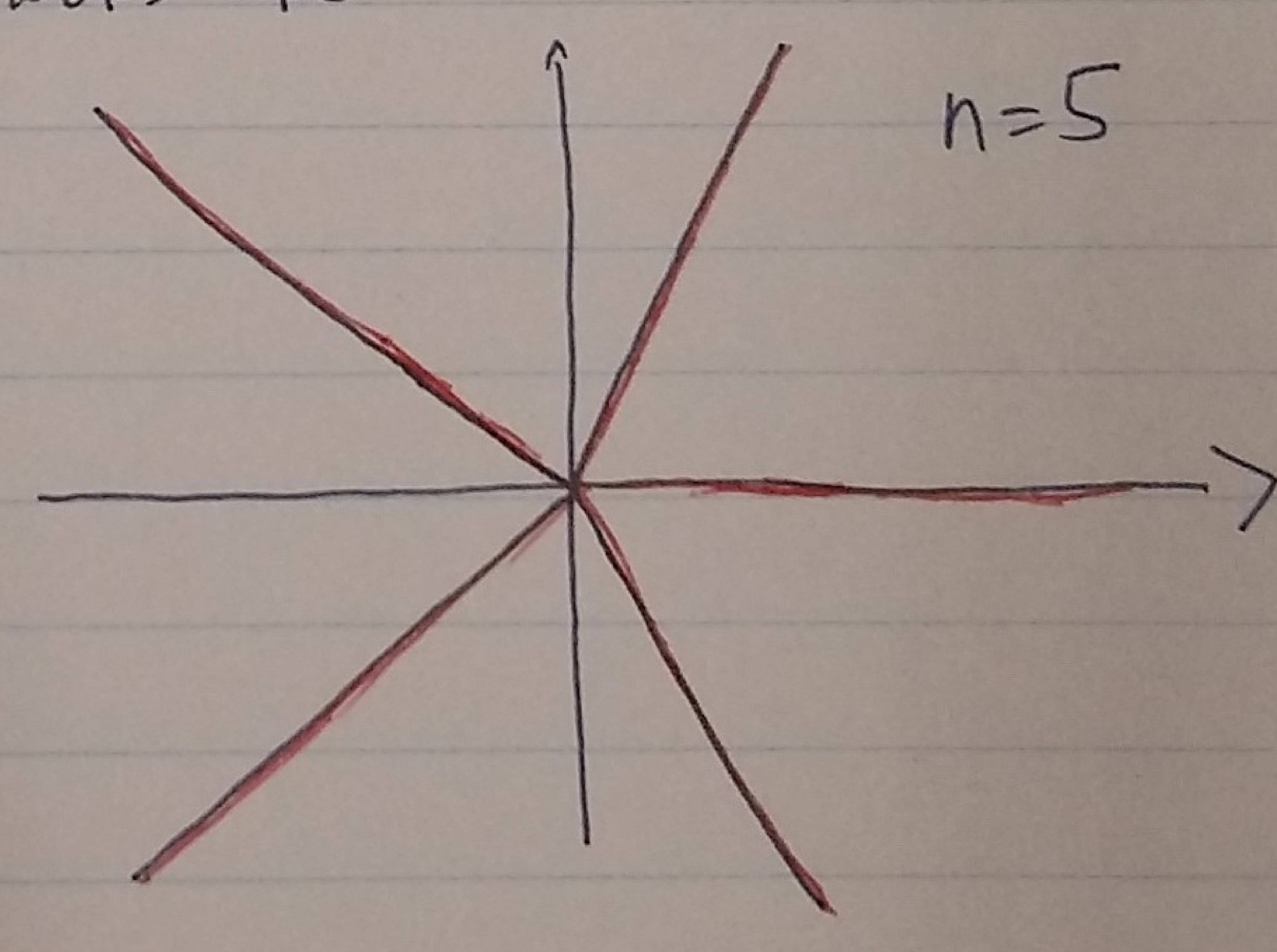
Ans: Write $z = re^{i\theta}$.

Then we have $r^n = r^n e^{in\theta}$

$$\Rightarrow \quad r^n(e^{in\theta}-1) = 0$$
$$\Rightarrow \quad r^n = 0 \quad \text{or} \quad e^{in\theta} = 1$$
$$\Rightarrow \quad r = 0 \quad \text{or} \quad \theta = \frac{2k\pi}{n},\ k = 0, 1, \cdots, n-1.$$

Graphically, the solution set looks like:



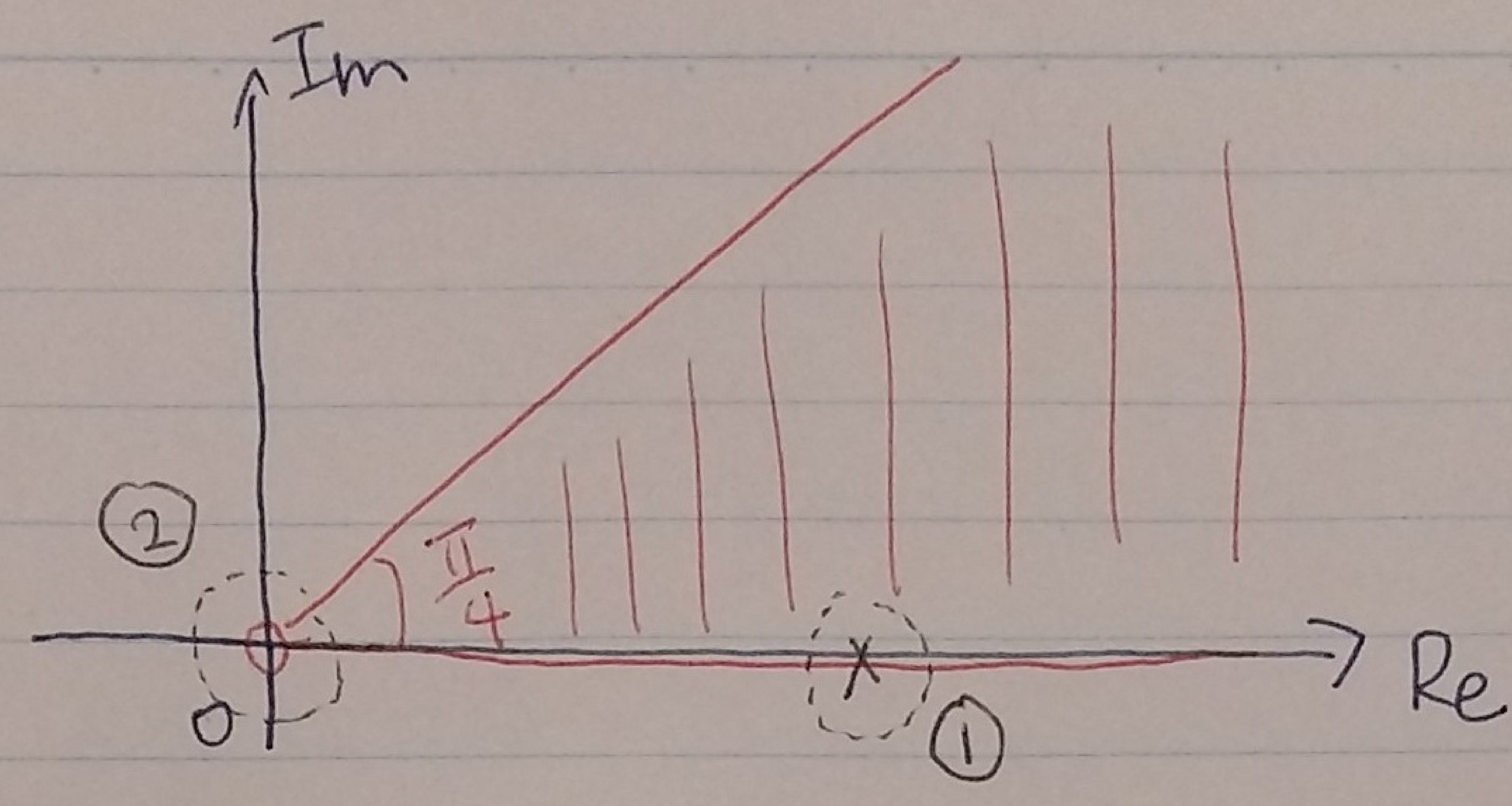


5) Sketch the set

$$\left\{ z \;\middle|\; 0 \leq \arg z \leq \frac{\pi}{4},\ z \neq 0 \right\}$$

Explain whether it is open / closed.

Ans:



It is not open because ① is not an interior point.
It is not closed because ② is an accumulative point which is not contained in the set.

Extra question: From 5), we see that a set can be neither open nor closed. It is natural to ask:

Q: Is there any set $\phi \neq S \subseteq \mathbb{C}$ which is both open and closed? Can we classify all of them?